
Majority and Positional Voting in a Probabilistic Framework
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Majority and Positional Voting in a Probabilistic Framework

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1. INTRODUCTION

A lot of attention has been devoted recently to the study of social decision-making procedures which combine voting with chance (for different approaches to the subject, see Zeckhauser (1969), Fishburn (1972*a*, *b*, 1978) Intriligator (1973) and Barberá and Sonnenschein (1977).) Decision schemes are procedures of this kind which assign a lottery on the set of alternatives to each N-tuple of rankings of alternatives. It is interpreted that they specify the probability with which each of the alternatives open to society is to be chosen, on the basis of the ordinal preferences on these alternatives expressed by the members of this society.

In this paper I define two wide classes of decision schemes—supporting size and point voting decision schemes—which can be viewed as natural adaptations to the probabilistic framework of two basic principles used in making deterministic choices: majority and positional voting. It is noted that these two principles, which are in general incompatible within a deterministic framework, can be jointly satisfied within the setting of decision schemes by what I call simple decision schemes, a third class which is the intersection of the two above. The theorems in the paper characterize these three classes in terms of the properties they satisfy. Some of these properties are, in turn, adaptations to the new framework of standard conditions in social choice theory: anonymity, neutrality and strategy-proofness. Two other new properties are introduced, which I call alternative independence and individual independence.

The following results are obtained:

> A decision scheme is a point-voting decision scheme if and only if it is anonymous, neutral, strategy-proof and individual independent.
> A decision scheme is a supporting size decision scheme if and only if it is anonymous, neutral, strategy-proof and alternative independent.
> A decision scheme is a simple decision scheme if and only if it is anonymous, neutral, strategy-proof, individual independent and alternative independent.

It may be worth relating the present results with Gibbard's characterization of strategy-proof decision schems. Gibbard (1977) proves that all such decision schemes can be expressed as the probability mixture of decision schemes, each of which is either unilateral or duple (where a decision scheme is unilateral if a single individual is the sole determiner of the probabilities assigned to alternatives, and duple if only two alternatives ever get a positive probability of being chosen). This has often been taken to be a very negative result, and yet point voting, supporting size and simple decision schemes are all strategy-proof. Is there a contradiction here? Certainly not a formal one: the latter are subclasses of the class characterized by Gibbard, and my proofs too rely on Gibbard's. The question may be more one of interpretation. Unilateral (resp. duple) decision schemes are certainly unattractive, because they treat individuals (resp. alternatives) in a discriminatory way. Yet, the probability mixtures of such schemes need not be themselves discriminatory. The results presented here show that there are indeed anonymous and neutral strategy-proof decision schemes; that is, schemes that are " nice " with regard to the same equal-treatment requirements which would make unilateral and duple schemes unacceptable. And, moreover, that all such decision schemes take very natural forms: they are not complicated to describe and to operate, and they are based upon the most widely accepted principles for social choice. This does not mean that these procedures are not flawed; if they are, though, it is not because they can be decomposed into unattractive components. Rather, it is because while meeting the interesting conditions pointed out here, they are not able to satisfy some others. On this, see Zeckhauser (1973), Barberá (1977*a*, *b*) and Gibbard (1977, Section 5). But then, finding out about such trade-offs, and seeing what can be done within the restrictions they impose, are natural tasks for social choice theory.

2. NOTATION AND DEFINITIONS[1]

Let V be a finite set, called the set of *alternatives*. Elements of V are denoted by x, y, z, w, ... M denotes the cardinality of V.

A *ranking*[2] of V is a binary relation P which, for all x and y, satisfies:

(i) Connectedness: $x \neq y \rightarrow (xPy \vee yPx)$

(ii) Asymmetry: $xPy \rightarrow \sim yPx$

(iii) Transitivity: $(xPy \;\&\; yPz) \rightarrow xPz$.

Let $I = \{1, 2, ..., N\}$ be an initial segment of the positive integers. I is called the *set of individuals*, and N is the *number of individuals*.

A *ranking N-tuple* over V is an N-tuple $(P_1, ..., P_N)$ of rankings of V. Ranking N-tuples are denoted by $\boldsymbol{P}$, $\boldsymbol{P}'$, $\boldsymbol{P}^*$, ... Ranking N-tuples are interpreted as functions which assign to each individual $i \in \boldsymbol{I}$ the ranking P_i of V.

A *measure* over V as a function m which assigns a non-negative number $m(x)$ to each member x of V. The sum $\sum_{x \in V} m(x)$ is called the *weight* of the measure. A *lottery* is a measure of weight one. Lotteries will be denoted by l.

A *scheme* for $\{I, V\}$ is a function from the set of ranking N-tuples over V to the set of measures over V which have a fixed weight α. The value α is called the weight of the scheme. Schemes will be denoted by f, f', ...

A *decision scheme* is a scheme of weight one. Decision schemes will be denoted by $\boldsymbol{d}$, d', ...

A decision scheme is, thus, a function which determines the probability with which each of the alternatives in V is to be selected, given the preferences of individuals expressed by a ranking N-tuple. The value of a decision scheme d at $\boldsymbol{P}$ is written $d\boldsymbol{P}$, and the probability that $d\boldsymbol{P}$ assigns to an alternative x is denoted by $d(x, \boldsymbol{P})$.

3. THREE CLASSES OF SCHEMES

Three specific classes of schemes are now defined. The first two of them—supporting size and point voting schemes—can be viewed as the probabilistic counterpart of well-known types of deterministic social choice procedures. The third class—simple schemes—will prove able to encompass the spirit of the two preceding ones.

The *rank* of an alternative x for an individual i within a ranking N-tuple $\boldsymbol{P}$ is defined as

$$r(i, x, \boldsymbol{P}) = \#\{y \mid y \in V \;\&\; yP_i x\} + 1.$$

Clearly, $1 \leqq r(i, x, \boldsymbol{P}) \leqq M$, for all x, i and $\boldsymbol{P}$.

The *supporting size* for an alternative x over an alternative y within a ranking N-tuple $\boldsymbol{P}$ is defined as

$$s(x, y, \boldsymbol{P}) = \#\{i \mid i \in I \;\&\; xP_i y\}.$$

Clearly, $0 \leqq s(x, y, \boldsymbol{P}) \leqq N$ and $s(x, y, \boldsymbol{P})+s(y, x, \boldsymbol{P}) = N$ for all x, y and $\boldsymbol{P}$.

A scheme f for (I, V) is a *point voting scheme* if there exists an M-dimensional vector of real numbers $(a_1, \ldots, a_M)$, to be called a *positional scoring vector*, such that

(i) $a_1 \geqq a_2 \geqq \ldots \geqq a_M \geqq 0$

(ii) for all $\boldsymbol{P}$ and $x, f(x, \boldsymbol{P}) = \sum_{i \in I} a_{r(i, x, \boldsymbol{P})}$.

It will be said, in this case, that f is *representable* as a point voting scheme with positional scoring vector $(a_1, \ldots, a_M)$.

Clearly, if d is a point voting decision scheme, it must be that $\sum_{i \in \{1, \ldots, M\}} a_i = 1/N$.

Point voting schemes operate in the following way: each alternative is given a score a_i every time that an individual ranks it in ith position. The total weight assigned to an alternative for a given ranking N-tuple is the sum of the scores that it has obtained on the basis of the preferences of each individual. In the case of point voting *decision* schemes the positional scoring vector is chosen so that the weights assigned to the different alternatives are probability distributions over V.

A scheme f for (I, V) is a *supporting size scheme* iff there exists an $N+1$-dimensional vector of real numbers $(b_N, \ldots, b_0)$, to be called the *supporting size scoring vector*, such that

(i) $b_N \geqq b_{N-1} \geqq \ldots \geqq b_0 \geqq 0$

(ii) $\exists K$ such that $b_j + b_{N-j} = K$ for all $j \leqq N/2$

(iii) for all $\boldsymbol{P}$ and $x, f(x, \boldsymbol{P}) = \sum_{z \in V-\{x\}} b_{s(x, z, \boldsymbol{P})}$.

We say then that f is representable as a supporting size scheme with supporting size scoring vector $(b_N, \ldots, b_0)$. Clearly, if d is a supporting size decision scheme it must be that, for $N/2 \leqq j \leqq N$, $b_j + b_{N-j} = 2/M(M-1)$.

Supporting size schemes operate on the basis of pairwise comparisons among alternatives. If j individuals prefer x to y, then x is assigned score b_j. The total weight assigned to an alternative is the sum of the scores it has obtained on the basis of pairwise comparisons with all others. In the case of supporting size *decision* schemes, the supporting size scoring vector is chosen so that the weights assigned to the different alternatives are probability distributions over V.

Point voting schemes can be seen as a natural probabilistic counterpart of a well-known type of deterministic rules for social choice, variously called scoring functions, representable voting functions, etc., and of which Borda's count is a classical example.

Supporting size schemes, on the other hand, convey the spirit of majority voting into the probabilistic framework. However, since decision schemes are more versatile than their deterministic counterparts, supporting size schemes are able to encompass, simultaneously, the features of simple and qualified majority voting.

Within a deterministic setting, the principles of majority and positional voting are incompatible when the number of alternatives is greater than two. This is not the case within the framework of decision schemes, since there may exist decision schemes which are representable both as point voting and supporting size schemes. The class of simple schemes, which is defined below, will prove to be the set of schemes which are representable both as supporting size and as point voting schemes.

Simple schemes assign a fixed weight to each alternative each time that it appears above some other in some individual ranking. The probability that they attach to each alternative is given by the weights that correspond to it on the basis of all possible pairwise comparisons with other alternatives over every individual, plus a constant term.

Given $x, y \in V$, $x \neq y$, and a ranking N-tuple $\boldsymbol{P}$, let

$$v_{xy}^i(\boldsymbol{P}) = \begin{cases} 1 & \text{if } xP_iy \\ 0 & \text{if } yP_ix \end{cases}$$

A scheme f is a *simple scheme* iff there exist numbers q and k such that

(i) $q \geqq 0$, $k \geqq 0$

(ii) for all $\boldsymbol{P}$ and x, $f(x, \boldsymbol{P}) = \sum_{y \in V-\{x\}} \sum_{i \in I} qv_{xy}^i(\boldsymbol{P}) + k$.

Clearly, if d is a simple decision scheme, it must be that $q = 2(1-Mk)/M(M-1)N$.

That some decision schemes can be represented in several forms is shown by the following example. Where $M = 3$ and $N = 4$, the same decision scheme can be represented as:

a supporting size decision scheme, with supporting size scoring vector (5/18, 4/18, 3/18, 2/18, 1/18),

a point voting decision scheme, with positional scoring vector (5/36, 3/36, 1/36), and

a simple decision scheme, with $q = 1/18$ and $k = 2/18$.

This example suggests a number of regularities in the general relationship among these three types of decision schemes, which are made explicit in the following proposition.

Proposition 1. *The set of simple decision schemes is the intersection of the sets of point voting and supporting size decision schemes. A decision scheme is representable as both a point voting and a supporting size decision scheme if and only if the components of the scoring vector in its representation in one of these forms constitute an arithmetic progression.*

The necessary elements to show this proposition are introduced in the rest of the paper; they are collected into a formal proof in the Appendix.

4. SOME PROPERTIES OF SCHEMES

We begin by defining three conditions which adapt to the probabilistic framework the notions of anonymity, neutrality and strategy-proofness.

A *permutation* is a one-to-one function from a finite set onto itself.

Given a permutation ρ on the set of alternatives and a preference ranking P, P^ρ is defined so that, for all $x, y \in V$, $xPy \leftrightarrow \rho(x)P^\rho\rho(y)$. Given ρ and an N-tuple of rankings $\boldsymbol{P} = (P_1, \ldots, P_N)$, $\boldsymbol{P}^\rho$ is defined to be the N-tuple $(P_1^\rho, \ldots, P_N^\rho)$.

Given a permutation σ on the set of individuals and an N-tuple of individual rankings $\boldsymbol{P} = (P_1, \ldots, P_N)$, $\boldsymbol{P}^\sigma$ is defined to be the N-tuple of rankings $\boldsymbol{P}^\sigma = (P_{\sigma^{-1}(1)}, \ldots, P_{\sigma^{-1}(N)})$.

A scheme f is *anonymous* iff, given any permutation σ on the set of individuals, $f(x, \boldsymbol{P}) = f(x, \boldsymbol{P}^\sigma)$ for all $x \in V$ and every N-tuple of rankings $\boldsymbol{P}$.

A scheme f is *neutral* iff, given any permutation ρ on the set of alternatives,

$$f(x, \boldsymbol{P}) = f(\rho(x), \boldsymbol{P}^\rho)$$

for all $x \in V$ and every N-tuple of rankings $\boldsymbol{P}$.

Thus, an anonymous scheme is one which does not discriminate among individuals, and a neutral scheme is one that does not discriminate among alternatives.

$\boldsymbol{P}$ and $\boldsymbol{P}'$ *agree off* k iff $(\forall i)$ $[(i \neq k) \rightarrow P_i = P_i']$.

$\boldsymbol{P}/_kP$ is the ranking N-tuple $\boldsymbol{P}'$ such that $P_k' = P$ and $\boldsymbol{P}'$ agrees with $\boldsymbol{P}$ off k.

A *utility scale* U over V is an assignment of real numbers to the members of V. Where U is a utility scale over V and l is a lottery over V, the *expected utility* $U(l)$ of l on scale U is defined by $U(l) = \sum_{x \in V} U(x).l(x)$.

Utility scale U *fits a ranking* P iff, for all $x, y \in V$,

$$U(x) > U(y) \leftrightarrow xPy.$$

A decision scheme d is *potentially manipulable* by k at $\boldsymbol{P}$ iff there are a utility scale U which fits P_k and a ranking P'_k of V such that $\boldsymbol{P}' = \boldsymbol{P}/_k P'_k$ and $U(d\boldsymbol{P}') > U(d\boldsymbol{P})$.

d is *manipulable* iff there are an individual k and a ranking N-tuple $\boldsymbol{P}$ such that d is potentially manipulable by k at $\boldsymbol{P}$. Otherwise, d is *strategy-proof.*

If d is potentially manipulable and individual k is endowed with utility scale U, he has an incentive to find out about the preferences that other individuals will declare and, eventually, to misrepresent his preferences when by doing so he is able to change the social ranking N-tuple from $\boldsymbol{P}$ to $\boldsymbol{P}/_k P'_k$.

If, on the contrary, d is strategy-proof, then no single agent can ever find it advantageous to misrepresent his preferences, and there are no incentives for any individual to engage in non-cooperative strategic considerations.

5. ANONYMOUS, NEUTRAL AND STRATEGY-PROOF DECISION SCHEMES

A scheme f is a *probability mixture* of schemes $f_1, \ldots, f_m$ iff there is a sequence $\alpha_1, \ldots, \alpha_m$, where $\alpha_i \geqq 0$ for all $i \in \{1, \ldots, m\}$ such that, for every $\boldsymbol{P}$ and x,

$$f(x, \boldsymbol{P}) = \alpha_1 f_1(x, \boldsymbol{P}) + \ldots + \alpha_m f_m(x, \boldsymbol{P}).$$

Where f is such a probability mixture, I will write $f = \alpha_1 f_1 + \ldots + \alpha_m f_m = \sum_{i \in \{1, \ldots, m\}} \alpha_i f_i$. By definition, $f'' = f - f'$ iff $f = f' + f''$.

Clearly, if a decision scheme d is the probability mixture of decision schemes $d_1, \ldots, d_m$, it has to be that $\sum_{i \in \{1, \ldots, m\}} \alpha_i = 1$.

The starting point for the results to be presented here is given by the following theorem.

Theorem 1. *A decision scheme is anonymous, neutral and strategy-proof iff it is a probability mixture of a point voting and a supporting size decision scheme.*

A detailed proof of this can be found in Barberá (1978). I will provide here an outline of the proof, which in turn rests upon the already mentioned characterization of strategy-proof decision schemes due to Gibbard (1977). Let us start by some definitions. A set X of alternatives *heads* a ranking P_k if all alternatives in X are preferred to those in $V - X$ according to P_k. A decision scheme d is *localized* iff for any k, $\boldsymbol{P}$, P'_k and X such that X heads both P_k and P'_k, $\sum_{x \in X} d(x, \boldsymbol{P}/P'_k) = \sum_{x \in X} d(x, \boldsymbol{P})$.

A *switch* is a reversal of two adjacent alternatives in a ranking. A decision scheme is *non-perverse* if switching an alternative upward never decreases its probability. It is *unilateral* if a single individual is the sole determiner of the probabilities assigned to alternatives. It is *duple* if only two alternatives ever get a positive probability of being chosen.[3] Gibbard proves that a decision scheme is strategy-proof if and only if it is a probability mixture of decision schemes, each of which is non-perverse, localized and either unilateral or duple.

Let's now go to the proof of Theorem 1. If d is anonymous, neutral and strategy-proof, it can be decomposed in such a way that no individual has a unilateral component unless all others have an identical unilateral component with identical weights (by anonymity); moreover, the probabilities assigned to different alternatives under different profiles by these unilateral components should only depend on the place of the alternative in the ranking of the relevant individual (by neutrality). The equal-weights probability mixture of these identical unilateral components can be represented as a point voting decision scheme, whose positional scoring vector is given by the contribution of each individual to the probability of an alternative being chosen on the basis of this alternative's position in that individual's ranking. Similarly, no pair of alternatives will have a duple component in the decomposition of a neutral anonymous and strategy-proof decision scheme unless all other pairs of alternatives have an identical duple component with identical weights (by neutrality). The probabilities assigned by these duple components to each alternative in that pair under different profiles should only depend on the number of individuals who support one alternative over the other (by anonymity). The equal-weights probability mixture of these identical duple components can be represented as a supporting size decision scheme, with scoring vector determined by the contribution of given-size groups of individuals to the probability of an alternative being chosen when they support it over some alternative. This outlines the necessity part of the proof. As for sufficiency, it should be clear by now that point voting schemes, supporting size schemes and their probability mixtures can be decomposed into probability mixtures of unilateral and duple schemes. This suffices for strategy-proofness; their neutrality and anonymity are obvious.

Observe that the characterization of anonymous, neutral and strategy-proof decision schemes given by Theorem 1 need not lead to a unique representation. For instance, it is clear from the example in Section 3 that, where $M = 3$ and $N = 4$, and

d^s is the supporting size decision scheme with supporting size scoring vector (5/18, 4/18, 3/18, 2/18, 1/18), while

d^p is the point voting decision scheme with positional scoring vector (5/36, 3/36, 1/36), all probability mixtures in the form $\alpha d^s + (1-\alpha)d^p$ represent the same decision scheme.

In view of this, some new definitions will now be introduced, which will lead to a unique decomposition of neutral, anonymous and strategy-proof decision schemes. This decomposition will prove instrumental for the characterization of point voting, supporting size and simple decision schemes, a task to be undertaken in Section 7.

A pure point voting scheme is one whose positional scoring vector satisfies

(i) $a_M = 0$

(ii) for some j, $a_{j+1} - a_j \neq a_j - a_{j-1}$, or $(\forall j) a_j = 0$.

A pure supporting size scheme is one whose supporting size scoring vector $(b_N, \ldots, b_0)$ satisfies

(i) $b_0 = 0$

(ii) for some h, $b_{h+1} - b_h \neq b_h - b_{h-1}$, or $(\forall j) b_j = 0$.

Proposition 2. *Every point voting scheme can be uniquely decomposed into the sum of a pure point voting scheme and a simple scheme. Every supporting size scheme can be uniquely decomposed into the sum of a pure supporting size scheme and a simple scheme.*

Proof. Let f be any point voting scheme, and $a = (a_1, \ldots, a_M)$ its positional scoring vector. Let $a' = (a_1 - a_M, \ldots, a_{M-1} - a_M, 0) = (a'_1, \ldots, a'_{M-1}, a'_M)$. Let $\Delta = \min (a'_{j-1} - a'_j)$, and $a'' = (a'_1 - (M-1)\Delta, a'_2 - (M-2)\Delta, \ldots, a'_{M-1} - \Delta, 0)$.

Then, $f = f' + f''$, where

(i) f'' is a pure point voting scheme with positional scoring vector a''

(ii) f' is a simple scheme, with $q = \Delta$ and $k = Na_M$.

It suffices to note that, for all i, x and $\boldsymbol{P}$,

$$r(i, x, \boldsymbol{P}) = M - \textstyle\sum_{y \in V, y \neq x} v^i_{xy}(\boldsymbol{P}).$$

Thus,

$$f(x, \boldsymbol{P}) = \textstyle\sum_{i \in I} a_{r(i,x,\boldsymbol{P})} = \sum_{i \in I} (a''_{r(i,x,\boldsymbol{P})} + a_M + (M - r(i, x, \boldsymbol{P}))\Delta) = \sum_{i \in I} a''_{r(i,x,\boldsymbol{P})}$$
$$+ \textstyle\sum_{i \in I} a_M + \sum_{i \in I} \sum_{y \in V, y \neq x} v^i_{xy}(P) = f'(x, \boldsymbol{P}) + f''(x, \boldsymbol{P}).$$

The proof that the decomposition is unique is left to the reader.

Similarly, notice that $s(x, y, \boldsymbol{P}) = \sum_{i \in I} v_{xy}^i(\boldsymbol{P})$. If f is a supporting size scheme with supporting size vector $(b_N, \ldots, b_0)$, let $b' = b_N - b_0, \ldots, b_1 - b_0, 0)$, $\delta = \min (b'_h - b'_{h-1})$ and $b'' = (b'_N - N\delta, \ldots, b'_1 - \delta, 0)$.

Now for any x and $\boldsymbol{P}$,

$$f(x, \boldsymbol{P}) = \sum_{y \in V, y \neq x} b_{s(x, y, \boldsymbol{P})} = \sum_{y \in V, y \neq x} b''_{s(x, y, \boldsymbol{P})} + \sum_{y \in V, y \neq x} \sum_{i \in I} v_{xy}^i(\boldsymbol{P})\delta + (M-1)b_0.$$

Therefore, f can be decomposed as the sum of f' and f'', where f'' is a pure supporting size scheme with supporting size scoring vector b'' and f' is a simple scheme with $q = \delta$ and $k = (M-1)b_0$.

The following assertions will be of use in what follows:

The sum of a simple and a point voting scheme is a point voting scheme.
The sum of a simple and a supporting size scheme is a supporting size scheme.
The sum of two simple schemes is a simple scheme.

Now we can provide a partial restatement of Theorem 1 in terms of a unique decomposition.

Theorem 1′. *Every neutral, anonymous and strategy-proof decision scheme can be uniquely decomposed into the sum of a pure point voting scheme, a pure supporting size scheme and a simple scheme.*

Proof. Let d be neutral, anonymous and strategy-proof. There will exist $0 \leqq \alpha \leqq 1$, d^p and d^s such that $d = \alpha d^p + (1-\alpha)d^s$ (by Theorem 1), where d^p is a point voting decision scheme and d^s is a supporting size decision scheme. Where $d^p = d_1^p + d'$ and $d^s = d_1^s + d''$ are the unique decompositions of d^p and d^s, with d', d'' simple schemes, we have that $d = \alpha d_1^p + (1-\alpha)d_1^s + \bar{d}$, where $\bar{d} = \alpha d' + (1-\alpha)d''$ is a simple scheme, αd_1^p is a point voting scheme and $(1-\alpha)d_1^s$ is a supporting size scheme.

It is left to the reader to check that the decomposition must be unique, even if the initial decomposition need not be.

6. TWO INDEPENDENCE CONDITIONS

Two new conditions on schemes are now introduced which will prove instrumental in characterizing the classes of point voting, supporting size and simple decision schemes. The two are conditions of independence.

One could think of the probability assigned by a scheme to an alternative, say x, as the result of a number of influencing factors and speak about " the part of x's probability which can be attributed to y's position relative to x " or " the part of x's probability which can be attributed to the preferences of the ith individual ". In general, though, these expressions may not have any meaning. However, I will argue that there exist circumstances where they do. Assume, for a moment, that we consider cases where expressions like the above are well defined. Then, we could require that " for any two N-tuples $\boldsymbol{P}$ and $\boldsymbol{P}'$ which agree on the ranking of x with respect to y, the contribution to x's probability which can be attributed to y's position should be the same for $\boldsymbol{P}$ and $\boldsymbol{P}'$ ". The first condition to be defined, that of alternative independence, attempts to formalize this notion.

Similarly, one could think of a condition requiring independence among each individual's contribution to the probability of an alternative, so that " for any two ranking N-tuples $\boldsymbol{P}$ and $\boldsymbol{P}'$ which agree on an individual i's ranking, the contribution to the probability of any $x \in V$ which can be attributed to i be the same for $\boldsymbol{P}$ and $\boldsymbol{P}'$ ". This is the second notion to be formalized, by means of the condition of individual-independence.

What we need to do first is to identify the circumstances under which it makes sense to talk about some alternative or some individual's " contribution " to x's probability. In fact, rather than such " contribution " what I will refer to is the difference between the contributions for certain situations.

A *switch* is a reversal of two adjacent alternatives in a ranking. $xP_k!y$ will mean that x is immediately above y in ranking P_k. Where $xP_k!y$, P_k^{yx} is the ranking which switches x and y in P_k and permutes no other alternatives.

Consider two ranking N-tuples $\boldsymbol{P}$ and $\boldsymbol{P}'$ such that, for a given pair of alternatives, x and y,

(i) for all i, either $xP_i!y$ or $yP_i!x$

(ii) for any i, if $xP_i!y$ either $P_i = P_i'$ or $P_i' = P_i^{yx}$

if $yP_i!x$ either $P_i = P_i'$ or $P_i' = P_i^{xy}$

(iii) for all i and $z \notin \{x, y\}$, $r(i, z, \boldsymbol{P}) = r(i, z, \boldsymbol{P}')$.

We then say that $C(x, y, \boldsymbol{P}, \boldsymbol{P}') = f(x, \boldsymbol{P}') - f(x, \boldsymbol{P})$ is the *gain* in x's probability which is *imputable to* y due to the change in preference from $\boldsymbol{P}$ to $\boldsymbol{P}'$, under scheme f.

A scheme f is *alternative-independent* iff, given any two pairs of ranking N-tuples $\boldsymbol{P}, \boldsymbol{P}', \bar{\boldsymbol{P}}, \bar{\boldsymbol{P}}'$, and any two alternatives x, y such that

(i) Conditions (i), (ii) and (iii) above hold for $x, y, \boldsymbol{P}, \boldsymbol{P}'$ and also for $x, y, \bar{\boldsymbol{P}}, \bar{\boldsymbol{P}}'$

(ii) For all i, $xP_iy \leftrightarrow x\bar{P}_iy$ and $xP_i'y \leftrightarrow x\bar{P}_i'y$, we have that

$$C(x, y, \boldsymbol{P}, \boldsymbol{P}') = C(x, y, \bar{\boldsymbol{P}}, \bar{\boldsymbol{P}}').$$

Requirement (i) restricts the definition to sets of ranking N-tuples for which it makes sense to speak about the gain in x's probability which is imputable to y. Requirement (ii) specifies that the ranking of x relative to y should be the same in $\boldsymbol{P}$ as in $\bar{\boldsymbol{P}}$, and in $\boldsymbol{P}'$ as in $\bar{\boldsymbol{P}}'$, for all individuals. The condition of alternative-independence then states that, under the conditions above, the gain in x's probability from $\boldsymbol{P}$ to $\boldsymbol{P}'$ should be the same as the gain from $\bar{\boldsymbol{P}}$ to $\bar{\boldsymbol{P}}'$.

Consider two ranking N-tuples $\boldsymbol{P}, \boldsymbol{P}'$ such that $\boldsymbol{P}' = \boldsymbol{P}/_kP_k'$. We then say, for any $x \in V$, that $D(x, \boldsymbol{P}, \boldsymbol{P}') = f(x, \boldsymbol{P}') - f(x, \boldsymbol{P})$ is the *gain* in x's probability which is *imputable to* k due to the change in preferences from $\boldsymbol{P}$ to $\boldsymbol{P}'$, under scheme f.

Scheme f is *individual-independent* iff, for every k, given any two pairs of ranking N-tuples $\boldsymbol{P}, \boldsymbol{P}', \bar{\boldsymbol{P}}, \bar{\boldsymbol{P}}'$, such that

(i) $\boldsymbol{P}' = \boldsymbol{P}/_kP_k'$, $\bar{\boldsymbol{P}}' = \bar{\boldsymbol{P}}/_k\bar{P}_k'$

(ii) $P_k = \bar{P}_k$, $P_k' = \bar{P}_k'$,

we have that $D(x, \boldsymbol{P}, \boldsymbol{P}') = D(x, \bar{\boldsymbol{P}}, \bar{\boldsymbol{P}}')$ for all $x \in V$.

Requirement (i) restricts the definition to sets of ranking N-tuples for which it makes sense to speak about the gain in x's probability which is imputable to k. Requirement (ii) specifies that the change in preferences from P_k to P_k' should be the same as in the one from $\bar{P}_k$ to $\bar{P}_k'$. The condition of individual-independence then states that the gain in x's probability from $\boldsymbol{P}$ to $\boldsymbol{P}'$ should be the same as from $\bar{\boldsymbol{P}}$ to $\bar{\boldsymbol{P}}'$.

7. THE CHARACTERIZATION OF POINT VOTING, SUPPORTING SIZE AND SIMPLE DECISION SCHEMES

Theorem 2. *A decision scheme is a point voting decision scheme iff it is anonymous, neutral, strategy-proof and individual-independent.*

Proof. The proof that all point voting decision schemes satisfy the requirements above is left to the reader.

Suppose d is anonymous, neutral, strategy-proof and individual-independent. By Proposition 2, it can be expressed as the sum $d = f_1 + f_2 + f_3$, where f_1 is a pure point voting scheme, f_2 is a simple scheme and f_3 is a pure supporting size scheme. Let a trivial scheme be one which assigns zero weight to all alternatives for every ranking N-tuple.

If f_3 is trivial, the conclusion follows from the fact that the sum of a point voting and a simple scheme is a point voting scheme. Suppose, then, that f_3 is non-trivial. Let $(a_1, \ldots, a_M)$ be the positional scoring vector associated with f_1 and $(b_N, \ldots, b_0)$ be the supporting size scoring vector associated with f_3. Since f_3 is a pure supporting size scheme, there exists an h such that $b_{h+1} - b_h \neq b_h - b_{h-1}$.

For given x, y, let $\boldsymbol{P}$ be such that

$$\text{for } 1 \leqq i \leqq h-1 \quad r(i, x, \boldsymbol{P}) = k-1 \quad \& \quad r(i, y, \boldsymbol{P}) = k$$

$$\text{for } h \leqq j \leqq N \quad r(j, x, \boldsymbol{P}) = k \quad \& \quad r(j, y, \boldsymbol{P}) = k-1.$$

Let $\boldsymbol{P}' = \boldsymbol{P}/_h P_h^{xy}$, $\boldsymbol{P}'' = \boldsymbol{P}'/_{h+1} P_{h+1}^{xy}$, $\boldsymbol{P}''' = \boldsymbol{P}''/_h P_h^{yx}$. Notice that $\boldsymbol{P}'' = \boldsymbol{P}'''/_h P_h^{xy}$.

By definition,

$$s(x, y, \boldsymbol{P}) = h-1,\ s(x, y, \boldsymbol{P}') = h,\ s(x, y, \boldsymbol{P}'') = h+1 \text{ and } s(x, y, \boldsymbol{P}''') = h.$$

Thus, by individual independence, and given the relationship in which $\boldsymbol{P}$, $\boldsymbol{P}'$, $\boldsymbol{P}'''$ and $\boldsymbol{P}''$ stand, it should be true that

$$D(x, \boldsymbol{P}, \boldsymbol{P}') = D(x, \boldsymbol{P}''', \boldsymbol{P}'').$$

Yet,

$$D(x, \boldsymbol{P}, \boldsymbol{P}') = d(x, \boldsymbol{P}') - d(x, \boldsymbol{P}) = f_1(x, \boldsymbol{P}') - f_1(x, \boldsymbol{P}) + f_2(x, \boldsymbol{P}') - f_2(x, \boldsymbol{P}) + f_3(x, \boldsymbol{P}') - f_3(x, \boldsymbol{P}),$$

while

$$D(x, \boldsymbol{P}''', \boldsymbol{P}'') = f_1(x, \boldsymbol{P}'') - f_1(x, \boldsymbol{P}''') + f_2(x, \boldsymbol{P}'') - f_2(x, \boldsymbol{P}''') + f_3(x, \boldsymbol{P}'') - f_3(x, \boldsymbol{P}''').$$

We have, by construction,

$$f_1(x, \boldsymbol{P}') - f_1(x, \boldsymbol{P}) = f_1(x, \boldsymbol{P}'') - f_1(x, \boldsymbol{P}''') = a_{k-1} - a_k,$$

and

$$f_2(x, \boldsymbol{P}') - f_2(x, \boldsymbol{P}) = f_2(x, \boldsymbol{P}'') - f_2(x, \boldsymbol{P}''') = q$$

while

$$f_3(x, \boldsymbol{P}') - f_3(x, \boldsymbol{P}) = b_h - b_{h-1} \neq b_{h+1} - b_h = f_3(x, \boldsymbol{P}'') - f_3(x, \boldsymbol{P}'''),$$

thus contradicting the fact that $D(x, \boldsymbol{P}, \boldsymbol{P}') = D(x, \boldsymbol{P}''', \boldsymbol{P}'')$.

Theorem 3. *A decision scheme is a supporting size decision scheme iff it is anonymous, neutral, strategy-proof and alternative-independent.*

Proof. The proof that all supporting size decision schemes satisfy the requirements above is left to the reader.

Suppose d is anonymous, neutral, strategy-proof and alternative-independent. By Proposition 2, it can be expressed as the sum $d = f_1 + f_2 + f_3$, where f_1 is a pure point voting scheme, f_2 is a simple scheme and f_3 is a pure supporting size scheme.

If f_1 is trivial, the conclusion follows from the fact that the sum of a supporting size and a simple scheme is a supporting size scheme.

Suppose f_1 is non-trivial. Then, there must exist k such that $a_{k-1} - a_k \neq a_k - a_{k+1}$. Let $\boldsymbol{P}$ be such that, for some given t $(1 \leqq t \leqq N)$,

$$\text{for } 0 \leqq i \leqq t-1 \quad r_i(x, \boldsymbol{P}) = k-1 \quad \& \quad r_i(y, \boldsymbol{P}) = k$$

$$\text{for } t \leqq i \leqq N \quad r_i(x, \boldsymbol{P}) = k \quad \& \quad r_i(y, \boldsymbol{P}) = k-1$$

Let P' be such that

for $0 \leqq i \leqq t-1$ $\quad r_i(x, \boldsymbol{P}') = k \quad \& \quad r_i(y, \boldsymbol{P}') = k+1$

for $t \leqq i \leqq N$ $\quad r_i(x, \boldsymbol{P}') = k+1 \quad \& \quad r_i(y, \boldsymbol{P}') = k.$

Let

$$\boldsymbol{P}'' = \boldsymbol{P}/_t P_t^{xy} \quad \text{and} \quad \boldsymbol{P}''' = P'/_t P_t'^{xy}.$$

By alternative-independence, it should be true that

$$C(x, y, \boldsymbol{P}, \boldsymbol{P}'') = C(x, y, \boldsymbol{P}', \boldsymbol{P}''').$$

We have

$$C(x, y, \boldsymbol{P}, \boldsymbol{P}'') = d(x, \boldsymbol{P}) - d(x, \boldsymbol{P}'') = f_1(x, \boldsymbol{P}) - f_1(x, \boldsymbol{P}'') + f_2(x, \boldsymbol{P}) - f_2(x, \boldsymbol{P}'') + f_3(x, \boldsymbol{P}) - f_3(x, \boldsymbol{P}''),$$

$$C(x, y, \boldsymbol{P}', \boldsymbol{P}''') = f_1(x, \boldsymbol{P}') - f_1(x, \boldsymbol{P}''') + f_2(x, \boldsymbol{P}') - f_2(x, \boldsymbol{P}''') + f_3(x, \boldsymbol{P}') - f_3(x, \boldsymbol{P}''').$$

Yet,

$$f_3(x, \boldsymbol{P}) - f_3(x, \boldsymbol{P}'') = f_3(x, \boldsymbol{P}') - f_3(x, \boldsymbol{P}''') = b_{t-1} - b_t,$$

and

$$f_2(x, \boldsymbol{P}) - f_2(x, \boldsymbol{P}'') = f_2(x, \boldsymbol{P}') - f_2(x, \boldsymbol{P}''') = -q,$$

while

$$f_1(x, \boldsymbol{P}) - f_1(x, \boldsymbol{P}'') = a_{k-1} - a_k \neq a_k - a_{k+1} = f_1(x, \boldsymbol{P}') - f_1(x, \boldsymbol{P}''').$$

This would contradict the fact that $C(x, y, \boldsymbol{P}, \boldsymbol{P}'') = C(x, y, \boldsymbol{P}', \boldsymbol{P}''')$.

Theorem 4. *A decision scheme is neutral, anonymous, strategy-proof, alternative-independent and individual-independent iff it is a simple decision scheme.*

Proof. The proof that all simple decision schemes satisfy the conditions above is left to the reader.

Suppose d meets the expressed requirements. Then, it can be decomposed into a sum $d = f_1 + f_2 + f_3$, where f_1 is a pure point voting scheme, f_2 is a simple scheme and f_3 is a pure supporting size scheme. It is clear from the proof of Theorem 2 that f_3 should be trivial for d to be individual-independent. Similarly, from the proof of Theorem 3, f_1 should be trivial for d to be alternative-independent. Thus, it must be that $d = f_2$, a simple scheme.

APPENDIX

Proof of Proposition 1. The first part of the proposition follows immediately from Theorems 2, 3 and 4. To prove the second part, first notice that simple decision schemes are representable both as point voting and supporting size decision schemes, in such a way that the elements of the scoring vector for each representation form an arithmetic progression. This comes from the fact that, as noted in the proof of Proposition 2,

$$r(i, x, P) = M - \sum_{y \in V, y \neq x} v_{xy}^i(\boldsymbol{P})$$

and $s(x, y, \boldsymbol{P}) = \sum_{i \in I} v_{xy}^i(\boldsymbol{P})$. It follows from this that, given a simple scheme with parameters q and k, it can also be expressed as a point voting scheme with positional scoring vector elements given by $a_{M-h} = k/N + hq$ $(0 \leqq h \leqq M-1)$; and also as a supporting size decision scheme with scoring vector elements given by $b_h = k/(M-1) + hq$ $(0 \leqq h \leqq N)$.

Notice next that if a decision scheme is representable as a point voting decision scheme (resp., as a supporting size decision scheme) and the elements of its scoring vector form an arithmetic progression, then it can be represented as a simple decision scheme. This is clear from the construction used in the proof of Proposition 2, and the fact that constant schemes can always be represented as simple schemes.

Now, suppose d is a supporting size decision scheme. If the elements of its scoring vector form an arithmetic progression, d can be expressed as a simple decision scheme, and thus also as a point voting decision scheme with a scoring vector whose elements form an arithmetic progression. If, on the contrary, the elements of its scoring vector as a supporting size decision scheme do not form an arithmetic progression, then d cannot be represented as a point voting decision scheme. For, suppose it could. Then d would belong to the intersection of the sets of point voting and supporting size decision schemes and would thus be representable as a simple decision scheme. But in this case the elements of the scoring vector in its representation as a supporting size decision scheme would form an arithmetic progression, in contradiction with our hypothesis.

A parallel argument starting from a point voting decision scheme d' would complete the proof of the Proposition.

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NOTES

1. The notation used here follows that in Gibbard (1977) whenever possible.
2. Notice that rankings do not allow for indifference among alternatives. This allows for simpler descriptions of the rules and for shorter proofs, but is not a crucial assumption.
3. These definitions are kept informal, since they will only be used in the heuristic argument that follows.

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